

Entangling capabilities of symmetric two qubit gates

Swarnamala Sirsi, Veena Adiga[‡] and Subramanya Hegde

Yuvaraja's College, University of Mysore, Mysore-05, India

E-mail: vadiga11@gmail.com

Abstract. Our work addresses the problem of generating maximally entangled two spin-1/2 (qubit) symmetric states using NMR, NQR, Lipkin-Meshkov-Glick Hamiltonians. Time evolution of such Hamiltonians provides various logic gates which can be used for quantum processing tasks. Pairs of spin-1/2's have modeled a wide range of problems in physics. Here we are interested in two spin-1/2 symmetric states which belong to a subspace spanned by the angular momentum basis $\{|j=1, \mu\rangle; \mu = +1, 0, -1\}$. Our technique relies on the decomposition of a Hamiltonian in terms of SU(3) generators. In this context, we define a set of linearly independent, traceless, Hermitian operators which provides an alternate set of SU(n) generators. These matrices are constructed out of angular momentum operators $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$. We construct and study the properties of perfect entanglers acting on a symmetric subspace i.e., spin-1 operators that can generate maximally entangled states from some suitably chosen initial separable states in terms of their entangling power.

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[‡] Also at : St.Joseph's College(autonomous), Bengaluru -27, India

1. Introduction

In the last few years there has been considerable increase in experimental activity [1] aiming to create entangled quantum states which have potential applications in quantum information processing tasks. In practice, these states are created by some physical operations involving the interaction between several systems. Thus analyzing these operations with regard to the possibility of creating maximally entangled states from an initial unentangled one and characterization of entangling capabilities of quantum operators play an important role in quantum information theory. Spin-1/2 (two level) systems have modeled a wide range of problems in physics. Considering N spin-1/2's (N qubits) in the symmetric subspace – the set of those N -particle pure states that remain unchanged by permutations of individual particles [2, 3], we define a set of $(N + 1)^2$ linearly independent, experimentally realizable cartesian tensor operators which provide different logic gates for quantum computation. In the particular case of two spin-1/2's NMR, NQR provide "hardware" for realizable quantum computers which involve the study of time evolution of Hamiltonian which can also be time dependent. Since these two qubit symmetric gates are capable of producing entanglement, quantifying their entangling capability is very important. Makhlin [4] has analyzed nonlocal properties of two-qubit gates and also studied some basic properties of perfect entanglers which are defined as the unitary operators that can generate maximally entangled states from some suitably chosen separable states. Zanardi et al. [5] have explored the entangling power of quantum evolutions in terms of mean linear entropy produced when unitary operator acts on a given distribution of pure product states. Kraus and Cirac [6], Reza khani [7] have given the tools to find the best separable two qubit input orthonormal product states such that some given unitary transformation can create maximally entangled quantum states. The entangling capability of a unitary quantum gate can be quantified by its entangling power $e_p(U)$ [5]. Balakrishnan et al. [8] have derived $e_p(U)$ in terms of local invariant G_1 . In this paper, we show that the two qubit symmetric quantum gates expressed in terms of newly defined linearly independent cartesian tensor operators belong to the class of perfect entanglers which can generate maximally entangled states from some suitably chosen product states. Further we show that these symmetric two qubit gates belong to a family of special perfect entanglers under certain conditions. This is a very relevant problem not only from the theoretical point of view, but also from the experimental one.

In section 2, we define an alternate representation of $SU(n)$ generators using the well known spherical tensor operators. Explicit form of these generators in 3-dimensional representation is given in 2.1. Section 2.2 deals with algebra of these cartesian tensor operators. Section 3 deals with two qubit symmetric gates and their entangling power in terms of the invariant G_1 . In section 3.1, we identify the conditions under which the perfect entangler can be classified as special perfect entangler. Further the entangling property of Lipkin-Meshkov-Glick Hamiltonian is studied in the spin-1 subspace.

1.1. Symmetric states

Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to $(N + 1)$, when N qubits respect exchange symmetry. Such a Hilbert space is considered to be spanned by the eigen states $\{|j, m\rangle; -j \geq m \leq +j\}$ of angular momentum operators J^2 and J_z , where $j = \frac{N}{2}$. The corresponding density matrix gets transformed to a 3×3 block form in the symmetric subspace characterized by the maximal value of total angular momentum $j_{max} = 1$. The symmetric subspace provides a convenient, albeit idealized, computationally accessible class of spin states relevant to many experimental situations such as spin squeezing. Completely symmetric systems are experimentally interesting, largely because it is often easier to nonselectively address an entire ensemble of particles rather than individually address each member. Permutationally symmetric states are useful in a variety of quantum information processing tasks and a class of these states have recently been implemented experimentally [9, 10].

2. Alternative representation of $SU(n)$ generators

It is well known that any Hermitian operator for a spin j system is given by [11]

$$\mathcal{H}(\vec{J}) = \frac{1}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} h_q^k \tau_q^{k\dagger}(\vec{J}), \quad (1)$$

where τ_q^k 's (with $\tau_0^0 = I$, the identity operator) are irreducible spherical tensor operators of rank 'k' in the $2j+1$ dimension spin space with projection 'q' along the axis of quantization in the real 3-dimensional space. The τ_q^k 's satisfy the orthogonality relation

$$Tr(\tau_q^{k\dagger} \tau_{q'}^{k'}) = (2j+1) \delta_{kk'} \delta_{qq'}. \quad (2)$$

Here the normalization has been chosen so as to be in agreement with Madison convention [12]. The spherical tensor parameters h_q^k which characterize the given Hermitian operator \mathcal{H} are given by $h_q^k = Tr(\mathcal{H} \tau_q^k)$. Since \mathcal{H} is Hermitian and $\tau_q^{k\dagger} = (-1)^q \tau_{-q}^k$, h_q^k 's satisfy the condition $h_q^{k*} = (-1)^q h_{-q}^k$. The spherical tensor parameters h_q^k 's have simple transformation properties under co-ordinate rotation [13] in the 3-dimensional space. i.e.,

$$(h_q^k)^R = \sum_{q'=-k}^{+k} D_{q'q}^k(\alpha\beta\gamma) h_{q'}^k, \quad (3)$$

where $D_{q'q}^k(\alpha\beta\gamma)$ denote Wigner-D matrix parametrized by Euler angles $(\alpha\beta\gamma)$.

Following the well known Weyl construction [13] for τ_q^k in terms of angular momentum operators J_x , J_y and J_z , we have

$$\tau_q^k(\vec{J}) = \mathcal{N}_{kj} (\vec{J} \cdot \vec{\nabla})^k r^k Y_q^k(\hat{r}), \quad (4)$$

where

$$\mathcal{N}_{kj} = \frac{2^k}{k!} \sqrt{\frac{4\pi(2j-k)!(2j+1)}{(2j+k+1)!}}, \quad (5)$$

are the normalization factors and $Y_q^k(\hat{r})$ are the spherical harmonics. Under rotations τ_q^k 's transform according to Wigner-D matrices. i.e.,

$$(\tau_q^k(\vec{J}))^R = \sum_{k=0}^{2j} \sum_{q=-k}^{+k} D_{q'q}^k(\alpha\beta\gamma) \tau_{q'}^k(\vec{J}). \quad (6)$$

We now define a set of linearly independent, traceless (except $(T^0)_0^0$), orthonormal Hermitian basis matrices $(T^\alpha)_q^k$, where $\alpha = +, -, 0$, $k = 1 \dots 2j$, and $q = 1$ to $+k$ as follows:

$$(T^+)_q^k = \frac{\tau_q^k + (\tau_q^k)^\dagger}{\sqrt{2(2j+1)}}, \quad (7)$$

$$(T^-)_q^k = \frac{i(\tau_q^k - (\tau_q^k)^\dagger)}{\sqrt{2(2j+1)}}, \quad (8)$$

and

$$(T^0)_0^k = \frac{\tau_0^k}{\sqrt{2j+1}}. \quad (9)$$

Observe that these matrices satisfy the relation $Tr((T^\alpha)_q^k (T^\beta)_{q'}^{k'}) = \delta_{\alpha\beta} \delta_{kk'} \delta_{qq'}$. In our new representation the most general density matrix can be written as

$$\rho = (r^0)_0^0 (T^0)_0^0 + \sum_{k=1 \dots 2j} (r^0)_0^k (T^0)_0^k + \sum_{\alpha=+,-} \sum_{k=1 \dots 2j} \sum_{q=1 \dots k} (r^\alpha)_q^k (T^\alpha)_q^k \quad (10)$$

Apart from $(T^0)_0^0$ which is proportional to identity matrix, there are $2j$ diagonal matrices namely $(T^0)_0^k, k = 1 \dots 2j$ and the rest are off diagonal.

2.1. $SU(3)$ generators:

In the particular case of two qubit symmetric subspace, our set of basis matrices § can be obtained from Equation(7,8,9) as ||

$$\begin{aligned} M_0 &= \sqrt{\frac{2}{3}} \tau_0^0, & M_1 &= \frac{\tau_1^1 + \tau_1^{1\dagger}}{\sqrt{3}}, & M_2 &= \frac{i(\tau_1^1 - \tau_1^{1\dagger})}{\sqrt{3}}, \\ M_3 &= \sqrt{\frac{2}{3}} \tau_0^1, & M_4 &= \frac{i(\tau_2^2 - \tau_2^{2\dagger})}{\sqrt{3}}, & M_5 &= \frac{i(\tau_1^2 - \tau_1^{2\dagger})}{\sqrt{3}}, \\ M_6 &= \frac{\tau_1^2 + \tau_1^{2\dagger}}{\sqrt{3}}, & M_7 &= \frac{\tau_2^2 + \tau_2^{2\dagger}}{\sqrt{3}}, & M_8 &= \sqrt{\frac{2}{3}} \tau_0^2. \end{aligned} \quad (11)$$

§ The above matrices are equivalent to the set of matrices with different normalization defined by R.J.Morris [19].

|| We have used a different notation for the basis set for the sake of simplicity.

These operators are explicitly represented in $|1m\rangle$ basis where $m = 1, 0, -1$ as follows:

$$\begin{aligned}
M_0 &= \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad M_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
M_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad M_5 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\
M_6 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

The above matrices are normalized i.e., $Tr(M_k M_{k'}) = 2\delta_{kk'}$ and M_1, \dots, M_7 have eigen values 1, 0, -1. For a pair of spin 1/2's (two qubits), there are 4×4 linearly independent operators that close under mutual commutator brackets. The 16 linearly independent operators of a four-state system can be chosen in a variety of matrix representations. One such choice [20] is $\frac{\sigma}{2}$, $\frac{\tau}{2}$, and $\frac{\sigma}{2} \otimes \frac{\tau}{2}$ where σ and τ are the individual pauli matrices for two qubits. Together with the 4×4 unit matrix, these 16 operators can be used to construct any operator describing magnetic couplings between the spins as well as the coupling of each spin to an external field. The co-efficient of these operators may in general be functions of time and have physical significance. In NMR and quantum computation, the effect of such operator combinations on paired spins and the solutions of the corresponding Hamiltonians is considered. Our complete set of basis matrices which are 9 in number are unitarily equivalent to combinations of these operators. i.e.,

$$\begin{aligned}
\sqrt{\frac{2}{3}} O_1 &\longrightarrow M_0, \quad -(O_5 + O_9) \longrightarrow M_1, \quad (O_6 + O_{10}) \longrightarrow M_2, \\
(O_2 + O_3) &\longrightarrow M_3, \quad -2(O_{15} + O_{16}) \longrightarrow M_4, \quad 2(O_8 + O_{12}) \longrightarrow M_5, \\
-2(O_7 + O_{11}) &\longrightarrow M_6, \quad 2(O_{13} - O_{14}) \longrightarrow M_7, \quad \frac{2}{\sqrt{3}}(2O_4 - O_{13} - O_{14}) \longrightarrow M_8.
\end{aligned}$$

And the unitary transformation which takes an operator from qubit basis to angular momentum basis is

$$\mathcal{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

As $|1m\rangle$ basis is related to the qubit basis through $|11\rangle = |\uparrow\uparrow\rangle$, $|10\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$, and $|1-1\rangle = |\downarrow\downarrow\rangle$, the above 9 matrices in the qubit basis are realized as

$$M_0 = \sqrt{\frac{2}{3}}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) + \frac{1}{6}((|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)),$$

$$\begin{aligned}
M_1 &= -\frac{1}{2}(|\uparrow\uparrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |\uparrow\downarrow\rangle(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) \\
&\quad + |\downarrow\uparrow\rangle(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) + |\downarrow\downarrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)) , \\
M_2 &= \frac{i}{2}(|\uparrow\uparrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |\uparrow\downarrow\rangle(-\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) \\
&\quad + |\downarrow\uparrow\rangle(-\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) - |\downarrow\downarrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)) , \\
M_3 &= (|\uparrow\uparrow\rangle\langle\uparrow\uparrow|) - (|\downarrow\downarrow\rangle\langle\downarrow\downarrow|) , \\
M_4 &= i((|\downarrow\downarrow\rangle\langle\uparrow\uparrow|) - (|\uparrow\uparrow\rangle\langle\downarrow\downarrow|)) , \\
M_5 &= \frac{i}{2}(|\uparrow\uparrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) - |\uparrow\downarrow\rangle(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) \\
&\quad - |\downarrow\uparrow\rangle(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) + |\downarrow\downarrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)) , \\
M_6 &= \frac{1}{2}(-|\uparrow\uparrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |\uparrow\downarrow\rangle(-\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) \\
&\quad + |\downarrow\uparrow\rangle(-\langle\uparrow\uparrow| + \langle\downarrow\downarrow|) + |\downarrow\downarrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)) , \\
M_7 &= ((|\uparrow\uparrow\rangle\langle\downarrow\downarrow|) + (|\downarrow\downarrow\rangle\langle\uparrow\uparrow|)) , \\
M_8 &= \frac{1}{\sqrt{3}}((|\uparrow\uparrow\rangle\langle\uparrow\uparrow|) - |\uparrow\downarrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) - |\downarrow\uparrow\rangle(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) .
\end{aligned}$$

In this representation the most general spin-1 Hamiltonian can be written as

$$\mathcal{H}(t) = \frac{1}{2} \sum_{i=0}^8 h_k(t) M_k . \quad (12)$$

Here M_k 's in terms of angular momentum operators J_x, J_y, J_z are given by $M_1 = -(J_x)$, $M_2 = (J_y)$, $M_3 = (J_z)$, $M_4 = -(J_x J_y + J_y J_x)$, $M_5 = (J_y J_z + J_z J_y)$, $M_6 = -(J_x J_z + J_z J_x)$, $M_7 = (J_x^2 - J_y^2)$, $M_8 = (3J_z^2 - 2)$. Note that the expansion co-efficients $h_k = \text{Tr}(\mathcal{H} M_k)$ are real and hence they constitute an experimentally measurable set of parameters.

2.2. Algebra of Cartesian tensor Operators

Commutation and anticommutation relations between M_k 's are given in table I and II respectively.

Table I: Table of commutators $[M_k, M_{k'}]$, with operators M_k in the first column and $M_{k'}$

in the top row, each entry provides the commutator $[M_k, M_{k'}]$.

M_k	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
M_1	0	$-iM_3$	iM_2	$-iM_6$	$-a$	iM_4	iM_5	$\sqrt{3}iM_5$
M_2	iM_3	0	$-iM_1$	iM_5	$-iM_4$	b	iM_6	$-\sqrt{3}iM_6$
M_3	$-iM_2$	iM_1	0	$2iM_7$	iM_6	$-iM_5$	$-2iM_4$	0
M_4	iM_6	$-iM_5$	$-2iM_7$	0	iM_2	$-iM_1$	$2iM_3$	0
M_5	a	iM_4	$-iM_6$	$-iM_2$	0	iM_3	iM_1	$-\sqrt{3}iM_1$
M_6	$-iM_4$	$-b$	iM_5	iM_1	$-iM_3$	0	$-iM_2$	$\sqrt{3}iM_2$
M_7	$-iM_5$	$-iM_6$	$2iM_4$	$-2iM_3$	$-iM_1$	iM_2	0	0
M_8	$-\sqrt{3}iM_5$	$\sqrt{3}iM_6$	0	0	$\sqrt{3}iM_1$	$-\sqrt{3}iM_2$	0	0

(13)

where $a = i(\sqrt{3}M_8 + M_7)$, $b = i(\sqrt{3}M_8 - M_7)$. Here $[M_3, M_8]=0$, $[M_4, M_8]=0$, $[M_7, M_8]=0$. Also there are four vector triplets $[M_1, M_2, M_3]$, $[M_1, M_4, M_6]$, $[M_4, M_2, M_5]$, $[M_5, M_3, M_6]$, satisfying the relation $[M_i, M_j] = -i\epsilon_{ijk}M_k$ and one triplet $[M_4, M_3, M_7]$, satisfying $-2i\epsilon_{ijk}M_k$.

Table II: Table of anticommutators $\{M_k, M_{k'}\}$, with operators M_k in the first column and $M_{k'}$ in the top row, each entry provides the anticommutator $\{M_k, M_{k'}\}$.

M_k	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
M_1	A	M_4	M_6	M_2	0	M_3	M_1	$-\frac{1}{\sqrt{3}}M_1$
M_2	M_4	B	M_5	M_1	M_3	0	$-M_2$	$-\frac{1}{\sqrt{3}}M_2$
M_3	M_6	M_5	C	0	M_2	M_1	0	$\frac{2}{\sqrt{3}}M_3$
M_4	M_2	M_1	0	C	$-M_6$	$-M_5$	0	$\frac{2}{\sqrt{3}}M_4$
M_5	0	M_3	M_2	$-M_6$	A	$-M_4$	M_5	$-\frac{1}{\sqrt{3}}M_5$
M_6	M_3	0	M_1	$-M_5$	$-M_4$	B	$-M_6$	$-\frac{1}{\sqrt{3}}M_6$
M_7	M_1	$-M_2$	0	0	M_5	$-M_6$	C	$\frac{2}{\sqrt{3}}M_7$
M_8	$-\frac{1}{\sqrt{3}}M_1$	$-\frac{1}{\sqrt{3}}M_2$	$\frac{2}{\sqrt{3}}M_3$	$\frac{2}{\sqrt{3}}M_4$	$-\frac{1}{\sqrt{3}}M_5$	$-\frac{1}{\sqrt{3}}M_6$	$\frac{2}{\sqrt{3}}M_7$	D

(14)

$$A = 2\sqrt{\frac{2}{3}}M_0 + M_7 - \frac{1}{\sqrt{3}}M_8, B = 2\sqrt{\frac{2}{3}}M_0 - M_7 - \frac{1}{\sqrt{3}}M_8, C = 2\sqrt{\frac{2}{3}}M_0 + \frac{2}{\sqrt{3}}M_8, \\ D = 2\sqrt{\frac{2}{3}}M_0 - \frac{2}{\sqrt{3}}M_8.$$

The relationship between M_k 's, and the Gell-Mann matrices Λ'_k s $k=1\dots 8$ is given by

$$M_1 = -\frac{1}{\sqrt{2}}(\Lambda_1 + \Lambda_6), M_2 = \frac{1}{\sqrt{2}}(\Lambda_2 + \Lambda_7), M_3 = \frac{1}{2}\Lambda_3 + \frac{\sqrt{3}}{2}\Lambda_8, M_4 = -\Lambda_5,$$

$$M_5 = \frac{1}{\sqrt{2}}(\Lambda_2 - \Lambda_7), M_6 = \frac{1}{\sqrt{2}}(\Lambda_6 - \Lambda_1), M_7 = \Lambda_4, M_8 = \frac{\sqrt{3}}{2}\Lambda_3 - \frac{1}{2}\Lambda_8.$$

3. Two qubit symmetric gates

The algebraic study of M_k 's and the corresponding symmetric two qubit gates is important not only for understanding fundamental properties of coupled spins and of quantum circuits, but also to study possible experimental implementations in different physical systems. Hamiltonian evolution provides the hardware for quantum gates. i.e., the time evolution of the operators M_k 's provide various symmetric logic gates for quantum computation. The closed form expression for $e^{iM_k\theta}$ is given by $B_k = e^{iM_k\theta} = I + (\cos\theta - 1)M_k^2 + i\sin\theta M_k$. Here $k = 1 \dots 7$ and I is a 3×3 unit matrix. Following are the explicit forms of the gates B_k 's in the symmetric subspace:

$$B_1 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \frac{-i\sin\theta}{\sqrt{2}} & -\sin^2 \frac{\theta}{2} \\ \frac{-i\sin\theta}{\sqrt{2}} & \cos\theta & \frac{-i\sin\theta}{\sqrt{2}} \\ -\sin^2 \frac{\theta}{2} & \frac{-i\sin\theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix}, B_2 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \sin^2 \frac{\theta}{2} \\ \frac{-i\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \sin^2 \frac{\theta}{2} & \frac{-i\sin\theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, B_4 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, B_5 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & -\sin^2 \frac{\theta}{2} \\ \frac{-i\sin\theta}{\sqrt{2}} & \cos\theta & \frac{-i\sin\theta}{\sqrt{2}} \\ -\sin^2 \frac{\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix},$$

$$B_6 = \begin{pmatrix} \cos^2 \frac{\theta}{2} & \frac{-i\sin\theta}{\sqrt{2}} & \sin^2 \frac{\theta}{2} \\ \frac{-i\sin\theta}{\sqrt{2}} & \cos\theta & \frac{i\sin\theta}{\sqrt{2}} \\ \sin^2 \frac{\theta}{2} & \frac{i\sin\theta}{\sqrt{2}} & \cos^2 \frac{\theta}{2} \end{pmatrix}, B_7 = \begin{pmatrix} \cos\theta & 0 & i\sin\theta \\ 0 & 1 & 0 \\ i\sin\theta & 0 & \cos\theta \end{pmatrix}, B_8 = \begin{pmatrix} e^{\frac{i\theta}{\sqrt{3}}} & 0 & 0 \\ 0 & e^{\frac{-2i\theta}{\sqrt{3}}} & 0 \\ 0 & 0 & e^{\frac{i\theta}{\sqrt{3}}} \end{pmatrix}.$$

A useful property of a two qubit symmetric gate is its ability to produce a maximally entangled state from an unentangled one. This property is locally invariant. It is well known that perfect entanglers are those unitary operators that can generate maximally entangled states from some suitably chosen separable states. The entangling properties of quantum operators have already been discussed in the literature [5, 8, 14]. Here we calculate the entangling power of two qubit symmetric gates following the simplified expression given by Balakrishnan et al. [8] according to which the gate B is a perfect entangler if its entangling power, $e_p(B) = \frac{2}{9}(1 - |G_1|)$ has the range $\frac{1}{6} \leq e_p \leq \frac{2}{9}$.

The local invariant G_1 [Ref. [4] table II] in terms of symmetric, unitary matrix m is given by $G_1 = \frac{\text{tr}^2 m}{16 \det[B]}$. Here $m = B_B^T B_B$ where the gates in the Bell basis are given by $B_B = U B U^\dagger$. U is a transformation matrix given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -\sqrt{2}i & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ -i & 0 & i & 0 \end{pmatrix}$$

connecting the angular momentum basis $|11\rangle, |10\rangle, |1-1\rangle, |00\rangle$ to the Bell basis $\frac{(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)}{\sqrt{2}}, \frac{i(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)}{\sqrt{2}}, \frac{(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)}{\sqrt{2}}, \frac{i(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle)}{\sqrt{2}}$. The relation $e_p(B) = \frac{2}{9}(1 - |G_1|)$ implies that gates having the same $|G_1|$ must necessarily possess the same entangling power e_p .

It is obvious that B_1, B_2, B_3 do not produce entanglement as they represent rotations

which is a local unitary transformation. Note that $|G_1| = 1$ and $e_p = 0$ for the above gates. Interestingly, for the gates B_4 , B_5 , B_6 and B_7 , $|G_1| = \cos^4(\theta)$. Observe that since $0 \leq G_1 \leq 1$ for $0 \leq \theta \leq \frac{\pi}{2}$, it is clear that $0 \leq e_p(B_B)_k \leq \frac{2}{9}$ ($k = 4 \dots 7$). All these above mentioned gates are perfect entanglers for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. Similarly the gate B_8 will have maximum entangling power i.e., $e_p = 2/9$ when $\theta = \sqrt{3}\frac{\pi}{2}$.

As an example, consider the direct product state $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, of two spinors in the qubit basis.

$$\begin{aligned} |\psi_{12}\rangle &= \begin{pmatrix} \cos \frac{\alpha_1}{2} \\ \sin \frac{\alpha_1}{2} e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} \cos \frac{\alpha_2}{2} \\ \sin \frac{\alpha_2}{2} e^{i\phi_2} \end{pmatrix} \\ &= \begin{pmatrix} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \\ \cos \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} e^{i\phi_2} \\ \sin \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} e^{i\phi_1} \\ \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} e^{i(\phi_1+\phi_2)} \end{pmatrix}, \end{aligned}$$

$0 \leq \alpha_{1,2} \leq \pi$, $0 \leq \phi_{1,2} \leq 2\pi$. Note that a separable state in the symmetric subspace will have the form

$$|\psi_{12}\rangle_{sym} = \begin{pmatrix} \cos^2 \frac{\alpha}{2} \\ \sqrt{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} e^{i\phi} \\ \sin^2 \frac{\alpha}{2} e^{2i\phi} \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha$ and $\phi_1 = \phi_2 = \phi$.

It is a well known fact that for a pure state of two qubits $|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$, the expression for concurrence is $C(\psi) = 2|ad - bc|$ [15]. For a maximally entangled quantum state concurrence $C = 1$. It can be observed that under the action of the gates B_4 , B_7 and B_8 (with e_p being maximum i.e., $2/9$), $|\psi_{12}\rangle_{sym}$ will become maximally entangled state when $\alpha = \frac{\pi}{2}$. i.e.,

$$\begin{aligned} B_4|\psi_{12}\rangle_{sym} &\xrightarrow{\alpha=\frac{\pi}{2}} \begin{pmatrix} -\frac{1}{2}e^{2i\phi} \\ \frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{1}{2} \end{pmatrix}, B_7|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=\frac{\pi}{2}} \begin{pmatrix} \frac{i}{2}e^{2i\phi} \\ \frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{i}{2} \end{pmatrix}, \\ B_8|\psi_{12}\rangle_{sym} &\xrightarrow{\alpha=\frac{\pi}{2}} \begin{pmatrix} \frac{i}{2} \\ -\frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{i}{2}e^{2i\phi} \end{pmatrix}. \end{aligned}$$

or in the qubit basis

$$\begin{aligned} B_4|\psi_{12}\rangle_{sym} &\xrightarrow{\alpha=\frac{\pi}{2}} -\frac{1}{2}e^{2i\phi} |\uparrow\uparrow\rangle + \frac{1}{2}e^{i\phi} |\uparrow\downarrow\rangle + \frac{1}{2}e^{i\phi} |\downarrow\uparrow\rangle + \frac{1}{2} |\downarrow\downarrow\rangle, \\ B_7|\psi_{12}\rangle_{sym} &\xrightarrow{\alpha=\frac{\pi}{2}} \frac{i}{2}e^{2i\phi} |\uparrow\uparrow\rangle + \frac{1}{2}e^{i\phi} |\uparrow\downarrow\rangle + \frac{1}{2}e^{i\phi} |\downarrow\uparrow\rangle + \frac{i}{2} |\downarrow\downarrow\rangle, \\ B_8|\psi_{12}\rangle_{sym} &\xrightarrow{\alpha=\frac{\pi}{2}} -\frac{i}{2} |\uparrow\uparrow\rangle + \frac{1}{2}e^{i\phi} |\uparrow\downarrow\rangle + \frac{1}{2}e^{i\phi} |\downarrow\uparrow\rangle + \frac{i}{2}e^{2i\phi} |\downarrow\downarrow\rangle. \end{aligned}$$

Similarly, the gates B_5 , B_6 acting on the symmetric separable state transform it into maximally entangled one when $\alpha = 0, \pi$. For eg:

$$B_5|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}, B_6|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

$$B_5|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \frac{1}{2} |\uparrow\uparrow\rangle - \frac{1}{2} |\uparrow\downarrow\rangle - \frac{1}{2} |\downarrow\uparrow\rangle - \frac{1}{2} |\downarrow\downarrow\rangle.$$

$$B_6|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \frac{1}{2} |\uparrow\uparrow\rangle - \frac{i}{2} |\uparrow\downarrow\rangle - \frac{i}{2} |\downarrow\uparrow\rangle + \frac{1}{2} |\downarrow\downarrow\rangle.$$

It can be noted that the operators B_8 and B_4 produce spin squeezing resulting from a single axis twisting and two axis counter twisting respectively [16]. Also possibility of physical realization of these spin squeezing operators are given in Ref.[18].

3.1. Special perfect entanglers

Rezakhani [7] has analyzed the perfect entanglers and found that some of them have the unique property of maximally entangling a complete set of orthonormal product vectors. Such operators for which $e_p = \frac{2}{9}$ belong to a well known family of special perfect entanglers. A study of using such special perfect entanglers as the building blocks of the most efficient universal gate simulation is also given in ref.[7]. Let us now study the conditions under which the perfect entanglers can be classified as special perfect entanglers. When $e_p = \frac{2}{9}$, B_4, \dots, B_8 in the qubit basis are given by

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, B_5 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_6 = \frac{1}{2} \begin{pmatrix} 1 & -i & -i & 1 \\ -i & 1 & -1 & i \\ -i & -1 & 1 & i \\ 1 & i & i & 1 \end{pmatrix}, B_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, B_8 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Following Rezakhani [7], the most general separable basis (upto general phase factors for each vector) is

$$|\psi_1\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle),$$

$$|\psi_2\rangle = (-b^*|\uparrow\rangle + a^*|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle),$$

$$|\psi_3\rangle = (e|\uparrow\rangle + f|\downarrow\rangle) \otimes (-d^*|\uparrow\rangle + c^*|\downarrow\rangle),$$

$$|\psi_4\rangle = (-f^*|\uparrow\rangle + e^*|\downarrow\rangle) \otimes (-d^*|\uparrow\rangle + c^*|\downarrow\rangle),$$

where $|a|^2 + |b|^2 = |c|^2 + |d|^2 = |e|^2 + |f|^2 = 1$.

When the gates B_4 , B_7 and B_8 as perfect entanglers act on the state - say $|\psi_1\rangle$, we obtain

$$[B_{4,7,8}]|\psi_1\rangle = -bd|\uparrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bc|\downarrow\uparrow\rangle + ac|\downarrow\downarrow\rangle.$$

This state is maximally entangled if its concurrence, $C = 4|abcd| = 1$. Thus these two qubit symmetric gates transform the orthonormal states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$ into maximally entangled ones if $|abcd| = |cdef| = \frac{1}{4}$. Similarly, for the gates B_5 and B_6 , condition for finding a full set of orthonormal product states is $|(a^2 + b^2)(c^2 + d^2)| = |(e^2 + f^2)(c^2 + d^2)| = 1$.

Let us consider the example of Lipkin-Meshkov-Glick interaction Hamiltonian [17, 18] which is widely used in nuclear physics.

$$\mathcal{H}_L = \mathcal{G}_1(J_+^2 + J_-^2) + \mathcal{G}_2(J_+J_- + J_-J_+) . \quad (15)$$

Here \mathcal{G}_1 and \mathcal{G}_2 are the coupling constants. In terms of our operators M_k 's,

$$\mathcal{H}_L = \mathcal{G}'_1 M_7 + \mathcal{G}'_2 (\sqrt{8}M_0 - M_8) , \quad (16)$$

where $\mathcal{G}'_1 = 2\mathcal{G}_1$ and $\mathcal{G}'_2 = \frac{2}{\sqrt{3}}\mathcal{G}_2$. Since $[M_7, M_8] = 0$, we have

$$e^{iH_L t} = B_L = \begin{pmatrix} e^{\sqrt{3}i\beta}\cos\xi & 0 & ie^{\sqrt{3}i\beta}\sin\xi \\ 0 & e^{2\sqrt{3}i\beta} & 0 \\ ie^{\sqrt{3}i\beta}\cos\xi & 0 & e^{\sqrt{3}i\beta}\cos\xi \end{pmatrix},$$

in spin-1 subspace. Here $\xi = \mathcal{G}'_1 t$ and $\beta = \mathcal{G}'_2 t$ and $e_p = \frac{2}{9}$ for $2\mathcal{G}_2 t = \frac{\pi}{2} + 2\mathcal{G}_1 t$. Under the action of this gate (with $e_p = \frac{2}{9}$), the separable state $|\uparrow\uparrow\rangle(|\downarrow\downarrow\rangle)$ becomes entangled for all values of t except when $t = \frac{n\pi}{4\mathcal{G}_1}$; $n=0,1,2,\dots$ and maximally entangled when $4\mathcal{G}_1 t = (2n+1)\frac{\pi}{2}$. For eg.,

$$B_L|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \cos(2\mathcal{G}_1 t) |\uparrow\uparrow\rangle + i\sin(2\mathcal{G}_1 t) |\downarrow\downarrow\rangle .$$

4. Conclusion

In conclusion, we have constructed traceless, Hermitian and linearly independent set of basis matrices which provides an alternative representation of $SU(n)$ generators. We have considered unitary evolutions of two spin-1/2 states in angular momentum subspace ($j=1$) and constructed physically realizable logic gates using $(2j+1)$ dimensional representation of the above set of basis matrices. Entangling properties of these gates have been studied in terms of their entangling power e_p . e_p is found to be maximum $(2/9)$ for B_4, \dots, B_8 under certain conditions which is the signature for special perfect entanglers. These logic gates are obtained by the exponentiation of the quadratic form of angular momentum operators $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$. As an example we have taken the well known Lipkin-Meshkov-Glick Hamiltonian and studied its entangling properties in spin-1 subspace. Further, we have shown that precisely at what time the initial separable state becomes maximally entangled under the action of perfect entanglers which consists of one-axis twisting and two axis twisting Hamiltonians that produce spin squeezing.

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